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TWO-ON-ONE COMBAT GAMES

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Abstract—A two-on-one combat game is a dynamic encounter in which there are three participants. Each participant controls its motion in some state space and has its own target set. Two of the participants team up against the third and attempt to capture it. The latter maneuvers to avoid capture and also to capture both team members. This paper employs differential game theory and the theory of combat games, to set up a mathematical decision framework and develop solution methods based on a penalty function approach for two-on-one combat games.

1. INTRODUCTION

A two-on-one combat game is a dynamic encounter in which there are three participants. Each participant, or player, controls its own motion in some state space, and also has a target set. Any player that enters into the target set of another is said to have been captured at the instant of entry; it is considered disabled for the rest of the encounter. The players are divided into two opposing teams, one consisting of a single player u and the other of the two players v_1 and v_2. The combat encounter terminates once all the players in one team are captured. The team with at least one surviving player is the winner of the game. The aim of each team is to win the game at the least cost to itself; if it cannot avoid losing, then its aim is to inflict the maximum damage upon its adversary.

Such encounters model common tactical situations in combat between aircraft, tanks or helicopters and so on. Besides the derivation of tactics for given situations, the study of these encounters has important implications for military planning as it should generate answers to the perennial question of how much quality is needed to offset quantity. This paper aims to set up a decision framework for such encounters based upon differential game theory.

In many ways this work is a natural outgrowth of earlier work on combat games [1–4]. As analyzed in these references, a combat game is an encounter between two combatants u and v, both of whom have offensive capabilities modelled by target sets. The playing space is divided into four regions; a u-win region, a v-win region, a region of joint capture and one of draw. Two differential games with state constraints are defined: the u-game in which player u tries to drive player v into u's target set with player v's target set as a state constraint and the v-game which is the reverse. In the u-game, player u minimizes a cost functional and v maximizes it, while in the v-game the reverse is true. The subdivision of the playing space into regions and the determination of optimal strategies within the regions is based upon the solution of these two games. In [4], value and saddle point strategies (in the sense of Friedman [5]) are shown to exist for the above state constrained differential games, under suitable conditions. An interior penalty function approach is proposed for the computation of approximate optimal feedback strategies; this is shown to possess suitable approximation and convergence properties. In this paper, the two-on-one combat game (TOCG) is analyzed in a similar fashion by setting up several state constrained differential games.

In the context of pursuit–evasion, Hagedorn and Breakwell [6] have investigated the problem of two pursuers and one evader. Breakwell and Hagedorn [7] investigate the capture of two evaders in succession. These papers treat specific problems where the vehicles are modelled as having simple motion, that is they move at constant speed and have infinite turn capability. The analysis has also been motivated towards finding capture regions by means of barriers [8]. In this paper, the general problem of setting up a decision framework for two-on-one combat games is addressed. This will hopefully be a step towards the eventual solution of "practical" problems of this genre.

The paper is organized as follows: Section 2 presents an informal discussion of TOCG emphasizing the questions that arise; in Section 3, a mathematical framework in terms of state-constrained differential games is proposed; and Section 4 considers solution methods.

2. A DESCRIPTION OF TWO-ON-ONE COMBAT GAMES

A typical two-on-one combat situation is pictured in Fig. 1. There are three vehicles u, v_1 and v_2, represented as mass-points, moving in a plane at constant speed and bounded turn rates ("car" model [8]). Each vehicle has a fan-shaped target set centered along its velocity vector. The vehicle u is superior to the other two in terms of its maneuverability and/or its weaponry (its target set is larger). One vehicle is said to capture another when the latter enters the former's target set. Vehicle u, or player u aims to capture both of the other vehicles (or players). Players v_1 and v_2, on the other hand, maneuver to prevent this and try to capture u instead.

The above picture reveals several facts about two-on-one combat games. First: suppose u can capture v_1 and v_2 each in the absence of the other, this information does not imply anything for the two-on-one combat game in general. This can be shown to be so by simple counter-examples. Second: once u has captured either opponent, it is intuitively clear that the subsequent encounter is a combat game between the surviving players. The mathematical formulation of the game must take this into account, both in the equations of motion and in the definition of capture. Suppose that v_1 is the player that is captured by u first; also the u vs v_2 game solution is available—the region in state space where u can capture v_2 and the value V (some function of the state vector $\mathbf{x}$) of the game are known. Then, the original two-on-one combat game can be reduced to an equivalent combat game. This equivalent combat game terminates when u captures v_1: u plays so as to minimize a cost functional that is the sum of an integral cost upto capture and V; the state constraint is the union of the v_1 and v_2 target sets. The players v_1 and v_2 can be combined into a team or super player v. The player v plays to maximize u's cost functional subject to a state constraint which is that v_1 and v_2 must avoid capturing each other. Third: there are several different possible outcomes of the two-on-one combat game; these have to be carefully classified according to some stated player preferences and a decision framework has to be evolved. Suppose, for example, the team v can capture u only with the loss of one of its members, but can avoid capture by u indefinitely (a draw), then the team can opt for either of these alternatives depending on its preferences. If it is assumed that destruction of u is of paramount importance, then the team will opt for the first alternative, otherwise the second may be chosen.

In the following a mathematical framework for two-on-one combat games is formulated and the above intuitive ideas are made precise in a general framework.

3. MATHEMATICAL FORMULATION

The equations of motion of the three players u, v_1 and v_2 are, respectively,

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{f}_1(t, \mathbf{x}_1, \mathbf{u}), \tag{1}$$

$$\frac{d\mathbf{x}_2}{dt} = \mathbf{f}_2(t, \mathbf{x}_2, \mathbf{v}_1), \tag{2}$$

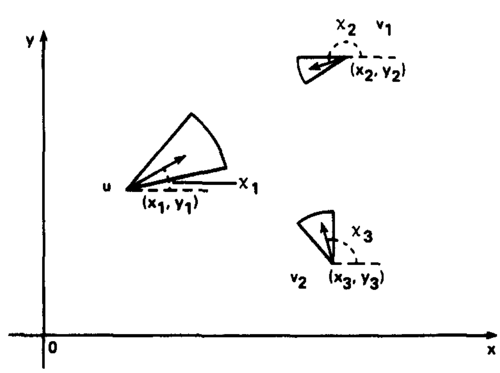

Fig. 1

$$\frac{d\mathbf{x}_3}{dt} = \mathbf{f}_3(t, \mathbf{x}_3, \mathbf{v}_2), \tag{3}$$

where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are n_1, n_2 and n_3 dimensional vectors respectively. The state vector of the game is the n-dimensional vector $\bar{\mathbf{x}}$ which has $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ as its components, that is $\bar{\mathbf{x}} := (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. The game starts at the instant t_0 with $\bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0$. The controls of the three players u, v_1 and v_2 are measurable functions taking values in $U \subset \mathscr{R}^p$, $V_1 \subset \mathscr{R}^{q_1}$, $V_2 \subset \mathscr{R}^{q_2}$, where U, V_1 and V_2 are compact subsets.

Each player has a target set in which it can capture any of the other two players. Player u's target set $\bar{\mathscr{T}}_u$ is given by

$$\bar{\mathscr{T}}_u := \bar{\mathscr{T}}_{u_1} \cup \bar{\mathscr{T}}_{u_2}, \tag{4}$$

where $\bar{\mathscr{T}}_{u_1} = G_{1,2} \times \mathscr{R}^{n_3}$, and $\bar{\mathscr{T}}_{u_2} = G_{1,3} \times \mathscr{R}^{n_2}$. Similarly, the target set of player v_1 is given by

$$\bar{\mathscr{T}}_{v_1} := \bar{\mathscr{T}}_{v_{11}} \cup \bar{\mathscr{T}}_{v_{12}}, \tag{5}$$

where $\bar{\mathscr{T}}_{v_{11}} = G_{2,1} \times \mathscr{R}^{n_3}$, and $\bar{\mathscr{T}}_{v_{12}} = G_{2,3} \times \mathscr{R}^{n_1}$; the target set of player v_2 is

$$\bar{\mathscr{T}}_{v_2} := \bar{\mathscr{T}}_{v_{21}} \cup \bar{\mathscr{T}}_{v_{22}}, \tag{6}$$

where $\bar{\mathscr{T}}_{v_{21}} = G_{3,1} \times \mathscr{R}^{n_2}$, and $\bar{\mathscr{T}}_{v_{22}} = G_{3,2} \times \mathscr{R}^{n_1}$.

In these expressions, the $G_{i,j} \subseteq \mathscr{R} \times \mathscr{R}^{n_i + n_j}$, for $i = 1, \ldots, 3, j = 1, \ldots, 3$ are closed subsets. It is assumed that there exists a time T^* such that $\forall t \geq T^*$, $[t, \bar{\mathbf{x}}] \in \bar{\mathscr{T}}_{u_1} \cap \bar{\mathscr{T}}_{u_2} \cap (\bar{\mathscr{T}}_{v_{11}} \cup \bar{\mathscr{T}}_{v_{21}})$.

The player i is said to have been captured by the player j at the instant $t_{i,j}$, $t_0 \leq t_{i,j} < T^*$, such that

$$t_{i,j} = \inf\{t > t_0 \,|\, [t, \bar{\mathbf{x}}(t)] \in \operatorname{int} \bar{\mathscr{T}}_j\}.$$

(Here int denotes the interior of the set, and inf denotes infimum.) The values of i,j range from 1 to 3, representing player u, v_1 and v_2 quantities respectively. For instance, $i = 2, j = 1$ in the above, means the capture of player v_1 by player u, so that $\bar{\mathscr{T}}_j = \bar{\mathscr{T}}_{u_1}$. Whenever player u is captured, the game terminates immediately. The game continues beyond the instant, say t_1, at which one of the players v_1 or v_2 is captured till another capture occurs or T^*, whichever is earlier. Over the time interval $[t_1, \dot{t}\,]$, the player captured at t_1, say v_1, is said to be disabled; its capture set becomes null. Also, once v_1 is captured, the sets $G_{1,2}$ and $G_{3,2}$ become equal to $\mathscr{R} \times \mathscr{R}^{n_1 + n_2}$ and $\mathscr{R} \times \mathscr{R}^{n_2 + n_3}$ respectively. If v_2 is the first to be captured, the corresponding subsets $G_{1,3}$ and $G_{2,3}$ expand to cover the respective spaces.

The TOCG terminates at the instant $\dot{t}$ at which one of the following events occurs:

(1) Player u is captured by the team v as follows:
   (a) at $\dot{t}$ either v_1 or v_2, or both jointly, capture u and no other capture occurs over $[t_0, \dot{t}\,]$;
   (b) at some instant $\hat{t}, \hat{t} \in [t_0, \dot{t}]$, player u captures either v_1 or v_2, and is captured by the survivor at $\dot{t}$;
   (c) at $\dot{t}$, player u and one team member, either v_1 or v_2, jointly capture each other while the other team member avoids capture over $[t_0, \dot{t}\,]$. In this event, the TOCG ends in a win for the team v.

(2) Both players v_1 and v_2 are captured by player u as follows:
   (a) at some instant $\hat{t}, \hat{t} \in [t_0, \dot{t}\,]$, player u captures either v_1 or v_2 and captures the survivor at $\dot{t}$. This event is a win of the TOCG for player u.

(3) The player u and the team v jointly capture each other as follows:
   (a) at some instant $\hat{t}, \hat{t} \in [t_0, \dot{t}\,]$, player u captures either v_1 or v_2; the surviving team member and u jointly capture each other at $\dot{t}$; here the TOCG ends in the joint capture of u by v and of v by u.

(4) The TOCG continues up to the time T^* as follows:
   (a) no player is captured up to T^*;
   (b) at some instant $\hat{t}, \hat{t} \in [t_0, \dot{t}\,]$, player u captures either v_1 or v_2; no other capture occurs up to T^*.

In describing all the possible terminations of the TOCG above, it is assumed that v_1 and v_2 play *sensibly* in that they do not run into each other's target sets.

From any given initial phase point $[t_0, \bar{\mathbf{x}}_0]$, one of the mutually exclusive events (1)–(4) must occur. First a system of player preferences for the above events that also distinguishes between the sub-events is set up. The preferences of player u are

$$2\mathrm{a} \succ 4\mathrm{b} \succ 4\mathrm{a} \succ 3\mathrm{a} \succ 1\mathrm{b},\, 1\mathrm{c} \succ 1\mathrm{a}, \tag{7}$$

where the symbol $\succ$ here denotes *is preferred to*. The preferences of team v are

$$1\mathrm{a} \succ 1\mathrm{b},\, 1\mathrm{c} \succ 4\mathrm{a} \succ 4\mathrm{b} \succ 3\mathrm{a} \succ 2\mathrm{a}. \tag{8}$$

In each ordering, the sub-events (1b) and (1c) are of equal preference, so that the player will distinguish between them only based upon a payoff.

Depending upon which of the four events occurs for each phase point, the phase space $\mathscr{R} \times \mathscr{R}^n$ can first be divided into the four regions Φ_u (u win), Φ_v (v win), $\Phi_{u \vee v}$ (draw), and $\Phi_{u \wedge v}$ (joint capture). These regions can be further subdivided into subregions where the sub-events corresponding to each event will occur. Since the game outcome depends entirely on the problem data, the initial point and the stated player preferences, all *consistent* strategies will be such that the game trajectory remains within the region to which the initial phasepoint belongs.

Next a set of differential games is defined; the solutions of these games are associated with the partition of the phase space into regions and subregions and will also yield a basis for determining player strategies. A u_1 game is defined as follows: a cost functional

$$J_{u_1} = g_u[\bar{\mathbf{x}}(\dot{t}_{u_1}), \dot{t}_{u_1}] + \int_{t_0}^{\dot{t}_{u_1}} h_u[t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{v}_1(t), \mathbf{v}_2(t)]\, \mathrm{d}t \tag{9}$$

is given with player u as the minimizer and players v_1, v_2 as the joint maximizers. The terminal time $\dot{t}_{u_1}$ is specified by

$$\dot{t}_{u_1} = \inf\{t \geqslant t_{3,1} \,|\, [t, \bar{\mathbf{x}}(t)] \in \operatorname{int} \bar{\mathscr{T}}_{u_1}\}, \tag{10}$$

where $t_{3,1}$, the time of capture of player v_2 by player u is given by

$$t_{3,1} = \inf\{t_0 \leqslant t < T^* \,|\, [t, \bar{\mathbf{x}}(t)] \in \operatorname{int} \bar{\mathscr{T}}_{u_2}\}. \tag{11}$$

This is subject to the following event constraints for player u:

$$\begin{aligned} &[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{11}} \cup \bar{\mathscr{T}}_{v_{21}} \quad \forall t_0 \leqslant t \leqslant t_{3,1}, \\ &[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{11}} \qquad \forall t_{3,1} < t \leqslant \bar{t}_{u_1}, \end{aligned} \tag{12}$$

and the constraint

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{12}} \cup \bar{\mathscr{T}}_{v_{22}} \quad \forall t_0 \leqslant t \leqslant t_{3,1}, \tag{13}$$

for player v. A u_2 game is defined with u minimizing a cost functional

$$J_{u_2} = g_u[\bar{\mathbf{x}}(\dot{t}_{u_2}), \dot{t}_{u_2}] + \int_{t_0}^{\dot{t}_{u_2}} h_u[t, \bar{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{v}_1(t), \mathbf{v}_2(t)]\, \mathrm{d}t, \tag{14}$$

and the players v_1, v_2 jointly maximizing it. [The functions g_u and h_u in expressions (9) and (14) are the same.] The terminal time $\dot{t}_{u_2}$ is specified by

$$\dot{t}_{u_2} = \inf\{t \geqslant t_{2,1} \,|\, [t, \bar{\mathbf{x}}(t)] \in \operatorname{int} \bar{\mathscr{T}}_{u_2}\}, \tag{15}$$

where $t_{2,1}$, the time of capture of player v_1 by player u is given by

$$t_{2,1} = \inf\{t_0 \leqslant t < T^* \,|\, [t, \bar{\mathbf{x}}(t)] \in \operatorname{int} \bar{\mathscr{T}}_{u_1}\}; \tag{16}$$

this is subject to the following event constraints for player u:

$$\begin{aligned} &[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{11}} \cup \bar{\mathscr{T}}_{v_{21}} \quad \forall t_0 \leqslant t \leqslant t_{2,1}, \\ &[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{21}} \qquad \forall t_{2,1} < t \leqslant \dot{t}_{u_2}, \end{aligned} \tag{17}$$

and the event constraint

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{12}} \cup \bar{\mathscr{T}}_{v_{22}} \quad \forall t_0 \leqslant t \leqslant t_{2,1} \tag{18}$$

for player v.

Next, three games are defined with the team v as the minimizer and player u as the maximizer. The first is the v game with a cost functional

$$J_v = g_v[\bar{\mathbf{x}}(\dot{t}_v), \dot{t}_v] + \int_{t_0}^{\dot{t}_v} h_v(t, \bar{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{v}_1(t), \mathbf{v}_2(t)] \, \mathrm{d}t. \tag{19}$$

The terminal time $\dot{t}_v$ is specified by

$$\dot{t}_v = \inf\{t > t_0 \,|\, [t, \bar{\mathbf{x}}(t)] \in \operatorname{int} \bar{\mathscr{T}}_{v_{11}} \cup \bar{\mathscr{T}}_{v_{21}}\}, \tag{20}$$

subject to the following event constraint for player v:

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_u \cup \bar{\mathscr{T}}_{v_{12}} \cup \bar{\mathscr{T}}_{v_{22}} \quad \forall t_0 \leqslant t \leqslant \dot{t}_v. \tag{21}$$

The second is the v_1 game with a cost functional

$$J_{v_1} = g_v[\bar{\mathbf{x}}(\dot{t}_{v_1}), \dot{t}_{v_1}] + \int_{t_0}^{\dot{t}_{v_1}} h_v[t, \bar{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{v}_1(t), \mathbf{v}_2(t)] \, \mathrm{d}t. \tag{22}$$

The terminal time $\dot{t}_{v_1}$ is specified by

$$\dot{t}_{v_1} = \inf\{t \geqslant t_{3,1} \,|\, [t, \bar{\mathbf{x}}(t)] \in \operatorname{int} \bar{\mathscr{T}}_{v_{11}}\}, \tag{23}$$

with $t_{3,1}$ once again given by equation (11). The event constraints that player v is subject to are

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{u_1} \quad \forall t_0 \leqslant t \leqslant \dot{t}_{v_1} \tag{24}$$

and

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{12}} \cup \bar{\mathscr{T}}_{v_{22}} \quad \forall t_0 \leqslant t \leqslant t_{3,1}. \tag{25}$$

Player u is subject to the event constraint

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{11}} \cup \bar{\mathscr{T}}_{v_{21}} \quad \forall t_0 \leqslant t \leqslant t_{3,1}. \tag{26}$$

The third is the v_2 game with a cost functional

$$J_{v_2} = g_v[\bar{\mathbf{x}}(\dot{t}_{v_2}), \dot{t}_{v_2}] + \int_{t_0}^{\dot{t}_{v_2}} h_v[t, \bar{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{v}_1(t), \mathbf{v}_2(t)] \, \mathrm{d}t. \tag{27}$$

The terminal time $\dot{t}_{v_2}$ is specified by

$$\dot{t}_{v_2} = \inf\{t \geqslant t_{2,1} \,|\, [t, \bar{\mathbf{x}}(t)] \in \operatorname{int} \bar{\mathscr{T}}_{v_{21}}\}, \tag{28}$$

with $t_{2,1}$ given by equation (16). The event constraints that player v is subject to are

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{u_2} \quad \forall t_0 \leqslant t \leqslant \dot{t}_{v_2} \tag{29}$$

and

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{12}} \cup \bar{\mathscr{T}}_{v_{22}} \quad \forall t_0 \leqslant t \leqslant t_{2,1}. \tag{30}$$

Player u is subject to the event constraint

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{11}} \cup \bar{\mathscr{T}}_{v_{21}} \quad \forall t_0 \leqslant t \leqslant t_{2,1}. \tag{31}$$

In all the v games, the functions g_v and h_v are the same.

Remark 3.1

Certain restrictions have to be imposed on the cost functionals in the above defined games for them to be physically meaningful. The function h_u in (9) and (14) should not depend on the disabled player's control or state vectors after the instant of capture; for example, in the u_1 game, h_u should not depend on $\mathbf{v}_2(t)$ or $\mathbf{x}_3(t)$ beyond the time instant $t_{3,1}$. Since the same function h_u appears also in the u_2 game and there it should not depend on $\mathbf{v}_1(t)$ or $\mathbf{x}_2(t)$ beyond the instant $t_{2,1}$, it follows that h_u should be independent of team v controls and state vectors after the capture of either v_1 or v_2. The function h_v in (19), (22) and (27) should be similarly restricted. It will be seen later in the paper (see Remark 3.2), that these restrictions on h_u and h_v are not as severely limiting as they may appear at first glance. □

Now suppose that the initial phasepoint $[t_0, \bar{x}_0] \in \Phi_u$. The v, v_1 and v_2 games are infeasible because team v cannot prevent player u from capturing both v_1 and v_2 and thus causing violation of the event constraints for v, given by equations (21), (24) and (29). The region Φ_u can be further subdivided into the three subregions Φ_u^1, Φ_u^2 and Φ_u^3, where the u_1 game is alone feasible, the u_2 game is alone feasible and both games are feasible, respectively. In Φ_u^3, player u will elect to play the u_1 game, if $J_{u_1} < J_{u_2}$ (both calculated with v playing its corresponding maximizing strategy), and the u_2 game if $J_{u_1} > J_{u_2}$. In the case of equality, an interesting possibility of nonuniqueness of strategies can arise. Thus, in Φ_u, strategies are computed by solving zero-sum differential games with event constraints.

If $[t_0, \bar{x}_0] \in \Phi_v$, both u_1 and u_2 games are infeasible. Team v's preferences, as given in (8), indicate that the team will choose to play the v game whenever it has feasible strategies for it, regardless of what payoff it can secure in the v_1 or v_2 games. Denote by Φ_v^1 the subset of Φ_v where the v game is feasible. In Φ_v^1, the v game, as given by equations (19)–(21) will be played. A further subdivision of Φ_v can be done by dividing $\Phi_v - \Phi_v^1$ into three subregions Φ_v^2, Φ_v^3 and Φ_v^4 where the v_1 game alone, the v_2 game alone and both games are feasible. In Φ_v^4, team v will select the game which will yield a lower payoff from among the v_1, v_2 games. In either case, player u chooses its strategy to maximize J_{v_1} or J_{v_2} as the case may be. Thus in Φ_v too, strategies are determined by solving zero-sum differential games with event constraints.

Next suppose that the initial phasepoint $[t_0, \bar{x}_0]$ is in the draw region $\Phi_{u\vee v}$. While both player u and team v prefer to draw if they cannot win, the former prefers outcome (4b) to outcome (4a), equation (7), and the latter the reverse, equation (8). Let $\Phi_{u\vee v}^1$ be the subset of $\Phi_{u\vee v}$ over which team v can enforce (4a). In this subregion, the v game is alone feasible and terminates at T^*. Thus, the strategies for u and v in this region are obtained by solving the v game with u maximizing J_v and v minimizing it subject to the event constraint (21); a fixed time zero-sum game situation.

The remainder of the $\Phi_{u\vee v}$ region, that is $\Phi_{u\vee v} - \Phi_{u\vee v}^1$, can be further subdivided into three subregions, $\Phi_{u\vee v}^2$, $\Phi_{u\vee v}^3$ and $\Phi_{u\vee v}^4$. In $\Phi_{u\vee v}^2$, player u can capture v_2 before time T^* but not v_1 so that the u_1 and v_1 games are feasible. Similarly, $\Phi_{u\vee v}^3$ is the region of feasibility of the u_2 and v_2 games with termination at T^*. The subregion $\Phi_{u\vee v}^4 = \Phi_{u\vee v} - \bigcup_{i=1}^3 \Phi_{u\vee v}^i$. In all of these subregions, termination is at the fixed time T^*.

In the region $\Phi_{u\vee v}^2$, player u will play to minimize J_{u_1} subject to the event constraint (12) and team v will play to minimize J_{v_1} subject to the event contraints (24) and (25). Similarly in $\Phi_{u\vee v}^3$, player u will play to minimize J_{u_2}, subject to the event constraint (17) and team v will play to maximize J_{v_2} subject to (29) and (30). These are a set of non-zero sum games.

In the region $\Phi_{u\vee v}^4$, a somewhat arbitrary decision scheme is proposed based upon two facts: first, the u_1 game and the v_2 game cannot be played together, neither can the u_2 and the v_1 games; second, player u can capture either v_1 or v_2 within T^* but not both. In this scheme, player u selects the game pair u_1, v_1 or u_2, v_2 that yields a lower payoff for it. Team v abides by u's choice and minimizes its corresponding payoff J_{v_1} (for the u_1, v_1 pair) or J_{v_2} (for the u_2, v_2 pair). Thus once again strategies are computed by solving non-zero sum differential games.

Finally suppose that $[t_0, \bar{x}_0] \in \Phi_{u\wedge v}$. The v game is not feasible. As before, $\Phi_{u\wedge v}^1$, $\Phi_{u\wedge v}^2$ and $\Phi_{u\wedge v}^3$ denote the subregions of $\Phi_{u\wedge v}$ where the game pair u_1, v_1 is feasible, u_2, v_2 is feasible and both pairs are feasible respectively. The terminal times $\dot{t}_{u_1} = \dot{t}_{v_1}$ and $\dot{t}_{u_2} = \dot{t}_{v_2}$. In $\Phi_{u\wedge v}^3$, the game pair which has the lower time will be played and capture will occur at the earliest time that each player can force. Each player will minimize its own cost functional. The resultant game is a non-zero sum game with event constraints.

Based upon the above analysis strategy selection rules for TOCG are given in Table 1. The TOCG models combat problems with a team of two players opposing one.

Remark 3.2

The scope of the restrictions imposed on the functions h_u and h_v (Remark 3.1) can now be discussed. The division of the region Φ_u into the subregions Φ_u^1, Φ_u^2 and Φ_u^3 depends only on the

Table 1. Strategy selection rules in TOCG

Subregion	Player u strategy	Team v strategy
Φ_u^1	$\min J_{u_1}$	$\max J_{u_1}$
Φ_u^2	$\min J_{u_2}$	$\max J_{u_2}$
Φ_u^3	$\min J_{u_1}$ or J_{u_2}†	$\max J_{u_1}$ or J_{u_2}†
Φ_v^1	$\max J_v$	$\min J_v$
Φ_v^2	$\max J_{v_1}$	$\min J_{v_1}$
Φ_v^3	$\max J_{v_2}$	$\min J_{v_2}$
Φ_v^4	$\max J_{v_1}$ or J_{v_2}‡	$\min J_{v_1}$ or J_{v_2}‡
$\Phi_{u \vee v}^1$	$\max J_v$	$\min J_v$
$\Phi_{u \vee v}^2$	$\min J_{u_1}$	$\min J_{v_1}$
$\Phi_{u \vee v}^3$	$\min J_{u_2}$	$\min J_{v_2}$
$\Phi_{u \vee v}^4$	$\min J_{u_1}$ or J_{u_2}§	$\min J_{v_1}$ or J_{v_2}§
$\Phi_{u \wedge v}^1$	$\min J_{u_1}$	$\min J_{v_1}$
$\Phi_{u \wedge v}^2$	$\min J_{u_2}$	$\min J_{v_2}$
$\Phi_{u \wedge v}^3$	$\min J_{u_1}$ or J_{u_2}§	$\min J_{v_1}$ or J_{v_2}§

†Whichever yields a lower minimum for player u.
‡Whichever yields a lower minimum for team v.
§Whichever yields a lower minimum for player u with team v minimizing the corresponding J_v.

feasibility of the games u_1 and u_2, and not on the cost functionals J_{u_1} or J_{u_2}. Thus in Φ_u^1, u can capture v_1 and v_2 only if it first captures v_2; in Φ_u^2 the reverse is true and in Φ_u^3 u can capture them in any order. The requirement that the function h_u in (9) and (14) be the same is needed only for $[t_0, \bar{\mathbf{x}}_0] \in \Phi_u^3$ to decide which of the two feasible games u_1 and u_2 is to be played. In this case, the restriction in Remark 3.1 is necessary. Suppose, however, that $[t_0, \bar{\mathbf{x}}_0] \in \Phi_u^1$. Then h_u in (9) need only be independent of $\mathbf{v}_2(t)$ and $\mathbf{x}_3(t)$ over $(t_{3,1}, \bar{t}_{u_1}]$. Similar remarks hold for the restrictions on the function h_v. □

Solution methods for TOCG are discussed next. These are based essentially on the relationship between TOCG and combat games.

4. SOLUTION METHODS

Once one of the players v_1 or v_2 is captured by u, the subsequent situation is clearly modelled by a combat game between the survivor and u, played in the space $\mathscr{R} \times \mathscr{R}^{n_1+n_3}$ (v_2 survives) or $\mathscr{R} \times \mathscr{R}^{n_1+n_2}$ (v_1 survives). Suppose the TOCG starts with $[t_0, \bar{\mathbf{x}}_0] \in \Phi_u^1$, so that the combat game is between u and v_1. Since u can win the u_1 game, it follows that u can win the combat game with the survivor v_1 once v_2 is captured, that is $[t_{3,1}^+, \mathbf{x}_1, \mathbf{x}_2] \in \phi_{u_1}$, where $\phi_{u_1} \subset \mathscr{R} \times \mathscr{R}^{n_1+n_2}$ is the u win region of the combat game between u and v_1 (called the u/v_1 game). Under the conditions (A1)–(A9), (B1)–(B3) of [4] the u/v_1 game has value, an X_u-saddle point [5] and approximate feedback strategies. Suppose that the value of this game is a continuous function $V_{u_1}(t, \mathbf{x}_1, \mathbf{x}_2)$, for all $[t, \mathbf{x}_1, \mathbf{x}_2] \in \phi_{u_1}$. This function can be simply extended to the space $\mathscr{R} \times \mathscr{R}^n$ by taking

$$\bar{V}_{u_1}[t, \bar{\mathbf{x}}] = \begin{cases} V_{u_1}(t, \mathbf{x}_1, \mathbf{x}_2), & \text{if } [t, \mathbf{x}_1, \mathbf{x}_2] \in \phi_{u_1}; \\ M, & \text{elsewhere,} \end{cases}$$

where M is a very large positive number. Next a u_1' game is defined with u minimizing a cost functional

$$J_{u_1'} = \bar{V}_{u_1}[t_{3,1}, \bar{\mathbf{x}}(t_{3,1})] + \int_{t_0}^{t_{3,1}} h_u[t, \bar{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{v}_1(t), \mathbf{v}_2(t)] \, \mathrm{d}t, \tag{32}$$

and with team v as the maximizer. The game terminates at the instant $t_{3,1}$ which is still defined by equation (11). This is subject to the event constraints

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{11}} \cup \bar{\mathscr{T}}_{v_{21}} \quad \forall t_0 \leqslant t \leqslant t_{3,1}, \tag{33}$$

for player u and (13) for team v.

The u_1' game is a combat game with an additional event constraint (13) for player v.

Strategies are defined for player u for all three games u_1, u_1' and u/v_1 as in [5] and are denoted by Δ_{u_1}, $\Delta_{u_1'}$ and Δ_{u/v_1} respectively. Team v is considered as a single *super player* with a control

$\mathbf{v}(t) = (\mathbf{v}_1(t), \mathbf{v}_2(t))$. Strategies are defined for team v for all three games u_1, u_1' and u/v_1 as in [5] and are denoted by Γ_{u_1}, $\Gamma_{u_1'}$ and Γ_{u/u_1} respectively.

The playing space of the u/v_1 game can be extended from $\mathscr{R} \times \mathscr{R}^{n_1+n_2}$ to $\mathscr{R} \times \mathscr{R}^n$ simply by adjoining the vector $\mathbf{x}_3$ to the state, with an arbitrary control function $\mathbf{v}_2'(t)$, say, chosen for the differential equation (3). This is possible because neither h_u nor the terminal instant $\bar{t}_{u_1}$ depend on either $\mathbf{x}_3$ or $\mathbf{v}_2(t)$. Hence, strategies Γ_{u/v_1} can be extended to strategies Γ'_{u/v_1} by defining the outcome $\mathbf{v}(t)$ of the latter to equal $(\mathbf{v}_1(t), \mathbf{v}_2(t))$.

It is assumed that h_u is positive and separable into terms that involve $\mathbf{u}(t)$ and $\mathbf{v}(t)$. Then, under conditions (A1)–(A9) of [4], suitably modified, Theorem 2.1 of [4] states that the u/v_1 game has both value and an X_u saddle point. Similarly, replacing condition (A4) of [4] by the requirement (A4′) $\bar{V}_{u_1}[t, \bar{\mathbf{x}}]$ is continuous on $[t, \mathbf{x}_1, \mathbf{x}_2] \in \phi_{u_1}$, the following theorem is obtained:

Theorem 4.1

If the conditions (A1)–(A3), (A4′), (A5)–(A9), all hold for the u_1' game given by equations (1)–(3), (32) and (33), and $[t_0, \mathbf{x}_0] \in \operatorname{int} \Phi_u^1$, then the u_1' game has both value and an X_u saddle point.

Proof. This theorem is essentially the same as Theorem 2.1 of [4], with the additional complication of the event constraint (13) for v. The assumed structure of the TOCG equations (1)–(3) ensures that u can in no way affect (13). Since team v is the maximizer in this game, the constraint (13) is taken into account by imposing a large negative penalty in cost for any strategy Γ_v' that leads to violation of (13)—see [5, Chapter 6]. Aside from this, the proof follows that of Theorem 2.1. □

Next, supposing the u/v_1 and the u_1' games have values and X_u saddle points and condition (A4′) holds, then the following theorem holds:

Theorem 4.2

If the u/v_1 game has value and an X_u saddle point (A4′) and Theorem 4.1 holds, then the u_1 game has value and an X_u saddle point. The value of the u_1 game equals that of the u_1' game for the same initial phase point $[t_0, \bar{\mathbf{x}}_0] \in \operatorname{int} \Phi_u^1$. □

Similar theorems can be stated for the u_2, v, v_1 and v_2 games. Next, an interior penalty function approach that yields arbitrarily close feedback approximations to the X_u saddle point strategies of the players is described.

4.1. Interior penalty function approach

The interior penalty function approach for solving combat games described in [4] can be easily extended to solving all the u and v games set up in Section 3. The penalty function is set up in exactly the same manner and is subject to the same conditions (B1)–(B3). For solving the u_1' game, the following sequence of unconstrained games

$$\bar{J}_{k,l} = J_{u_1'} + r_k J_{\mathrm{p}} - s_l J_{\mathrm{q}}, \tag{34}$$

is considered. Here $\{r_k\}$ and $\{s_l\}$ are decreasing sequences of positive numbers, J_{p} is an interior penalty term for the event constraint (33) and J_{q}, an interior penalty term for (13). As in [4], the functionals J_{p} and J_{q} approach infinity as the game trajectory approaches the boundary of the corresponding event constraint. The following two theorems are simply paraphrases of Theorems 3.2 and 3.4 of [4] for the u_1' game.

Theorem 4.3

Assume that conditions (A1)–(A3), (A4′), (A5)–(A9), (B1)–(B3) of [4] hold and let $[t_0, \bar{\mathbf{x}}_0] \in \operatorname{int} \Phi_u^1$, then the (unconstrained) differential game with cost functional (34), motion given by (1)–(3), termination defined by (11), has value, an X_u saddle point and optimal feedback strategies (almost everywhere), for any fixed constants r_k and s_l. □

Definition 4.4

For given r_k and s_l, the above unconstrained differential game is called the (u_1', k, l) penalty game. □

Theorem 4.5

Assume that conditions (A1)–(A3), (A4′), (A5)–(A9), (B1)–(B3) of [4] hold and let $[t_0, \bar{\mathbf{x}}_0] \in \operatorname{int} \Phi_u^1$; then for each $\epsilon > 0$, there exists a positive number K such that $\forall l, k \geqslant K$, the value of the (u_1', k, l) penalty game satisfies the inequality

$$J_{u_1'}(\Delta_{u_1'}^*, \Gamma_{u_1'}^*) < \bar{J}_{k,l}(\Delta_{k,l}^*, \Gamma_{k,l}^*) \leqslant J_{u_1'}(\Delta_{u_1'}^*, \Gamma_{u_1'}^*) + \epsilon, \tag{35}$$

where the asterisk indicates saddle point strategies. □

This justifies using the penalty function approach to compute optimal approximating strategies for the u_1' game for player u and team v. Similarly, Theorem 3.4 of [4] justifies penalty function computation of optimal approximating strategies for the u/v_1 game. These two results can be put together to give a penalty functional

$$\bar{J}_{j,k,l} = J_{u_1} + q_j J_0 + r_k J_{\mathrm{p}} - s_l J_{\mathrm{q}} \tag{36}$$

for the u_1 game. Here $\{q_j\}$ is a decreasing sequence of positive numbers and J_0 is a penalty functional for the event constraint

$$[t, \bar{\mathbf{x}}(t)] \notin \operatorname{int} \bar{\mathscr{T}}_{v_{11}} \quad \forall t_{3,1} < t \leqslant \bar{t}_{u_1}$$

for player u. Based upon Theorems 4.3–4.5, the following theorems extend the penalty function approach to the u_1 game:

Theorem 4.6

Assume that conditions (A1)–(A9), (B1)–(B3) of [4] hold and let $[t_0, \bar{\mathbf{x}}_0] \in \operatorname{int} \Phi_u^1$, then the (unconstrained) differential game with cost functional (36), motion given by (1)–(3), termination defined by (10), has value, an X_u saddle point and optimal feedback strategies (almost everywhere), for any fixed constants q_j, r_k and s_l. □

Definition 4.7

For given q_j, r_k and s_l, the above unconstrained differential game is called the (u_1, j, k, l) penalty game. □

Theorem 4.8

Assume that conditions (A1)–(A9), (B1)–(B3) of [4] hold and let $[t_0, \bar{\mathbf{x}}_0] \in \operatorname{int} \Phi_u^1$; then for each $\epsilon > 0$, there exists a positive number K such that $\forall j, k, l \geqslant K$, the value of the (u_1, j, k, l) penalty game satisfies the inequality

$$J_{u_1}(\Delta_{u_1}^*, \Gamma_{u_1}^*) < \bar{J}_{j,k,l}(\Delta_{j,k,l}^*, \Gamma_{j,k,l}^*) \leqslant J_{u_1}(\Delta_{u_1}^*, \Gamma_{u_1}^*) + \epsilon. \quad \square \tag{37}$$

The optimal feedback strategies for the (u_1, j, k, l) penalty game can be computed via the Isaacs equation [9] by any other computational technique. Theorem 4.8 shows that the value of the penalty game can be made to approach arbitrarily close to the u_1 game.

Similar results can be derived for the u_2, v, v_1 and v_2 games.

5. CONCLUDING REMARKS

A decision framework for TOCG has been developed in this paper. This is based upon the solution of five event-constrained differential games. An interior penalty function approach for solving these games is described.

The differential equations that describe the TOCG in Section 4 are sufficiently general to represent most tactical applications. In particular the example described in Section 3 fits the above description. An extended version of the Turret game [2] with three players is being analyzed.

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